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Infrared Finiteness in Yang-Mills Theories

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ABSTRACT

The infrared divergences of renormalizable theories with coupled massless fields (in particular the Yang-Mills theory) are shown to cancel for transition probabilities corresponding to finite energy resolution detectors, just as in quantum electrodynamics. This result is established through lowest non-trivial order in perturbation theory for the detection of massive muons in a QED theory containing massless electrons or the detection of massive quarks in a Yang-Mills theory.

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In quantum electrodynamics, physically sensible transition probabilities are infrared finite to all orders in α. It has been shown by many people[1,2] beginning with Bloch and Nordsieck that the infrared divergences associated with virtual corrections are cancelled by corresponding divergences in the emission of undetected photons whose total energy is less than the energy resolution ΔE of the detector. A central question in the study of Yang-Mills theories is whether an analog to the Bloch-Nordsieck program can be carried out. Our work suggests that this is possible for any renormalizable theory containing coupled massless fields along with massive ones. This result should help to sharpen questions about the confinement mechanism in gauge theories. In particular, it has been speculated that confinement is connected to the existence of mass singularities in perturbation theory. However, to lowest non-trivial order we have proven that there are no singularities in experimentally accessible transition probabilities. We conjecture that this is true to any finite order.

We begin by reviewing some known features of quantum electrodynamics in the limit $m_e \to 0$ since this model contains coupled massless fields, a feature shared by the Yang-Mills theory. Even with $m_e=0$, certain transition probabilities remain free of infrared singularities[3,4]. We then proceed to the Bloch-Nordsieck problem for two different theories involving massive fields in addition to coupled massless ones. The first (model 1) is quantum electrodynamics with a massive muon and a massless electron. The second (model 2) is the Yang-Mills theory with massive quarks coupled to the massless gauge fields. It is useful to study the Abelian model first since its infrared structure in perturbation theory is similar to the Yang-mills theory and yet it avoids some of the non-Abelian complexity. In either theory, the mass renormalization of the fermions is performed on the mass-shell and the resulting renormalized fermion mass μ is gauge independent. However in any theory with coupled massless fields it is necessary to define the coupling constant away from mass-shell to avoid introducing spurious infrared singularities[5]. For both models, we explicitly analyze production by an external local current $J_\mu(x)$ followed by the detection of a massive fermion with energy resolution ΔE. In the Abelian model, $J_\mu(x)=\bar{\mu}(x)\gamma_\mu\mu(x)$, where $\mu(x)$ is the muon field and the detector triggers on muon number. For the Yang-Mills model, $J_\mu(x)=\sum_i\bar{q}_i(x)\gamma_\mu q_i(x)$, a group (color) singlet. The quark detector is color-blind, triggering on, say, electric charge.

If the limit $m_e\to 0$ is taken in ordinary electrodynamics, a single electron state becomes degenerate in energy with a state consisting of an electron and any number of parallel moving photons and electron-positron pairs. Summation over these degenerate states is necessary for infrared finiteness. Consider, for example, the production of electrons, positrons and photons by an external current $j_\mu(x)=\bar{\psi}_e(x)\gamma_\mu\psi_e(x)$. Any Feynman diagram contributing to $\int d^4x\; e^{iq\cdot x}\langle 0|T(j_\mu(x)j_\nu(0))|0\rangle$ is finite in this limit since the external momentum provides an infrared cutoff[6]. Thus the sum of contributions to the total production cross section corresponding to the different cuts of each such diagram will be infrared finite in the limit $m_e\to 0$. Note that a partial cross section corresponding to the detection of a single massless electron with energy between E and E+ΔE is impossible to measure and is in fact logarithmically divergent[7].

The extension of this result to the Yang-Mills theory has been checked through low orders of perturbation theory[8]. The finiteness of the total cross section can be shown with massless or massive quarks coupled to the gluons. However, with massive quarks, the question of the finiteness of a partial cross section with finite energy resolution quark detectors (a Bloch-Nordsieck result) is less easily answered.

We consider this problem for both models 1 and 2. The computation is organized by grouping together the different unitarity cuts of each diagram contributing to $\Pi_{\mu\nu}(q)=\int d^4x\ e^{iq\cdot x}\langle 0|T\ J_\mu(x)J_\nu(0)|0\rangle$. The phase space integral over the massive fermion is however restricted by the detector kinematics. The object to be computed is the cross section for the inclusive detection of a massive fermion. We normalize by the Born cross section and define the dimensionless transition probability $R_{\Delta E}(E/M,\ \Delta E/M,\ \mu/M,\ m/M,\ g(M))$ where E represents the energies (center of mass, fermion energy), ΔE is the energy resolution[9], μ is the heavy fermion mass, m represents some infrared cutoff and g(M) is the renormalized coupling constant[5]. In the Abelian model the infrared cutoff m is the electron mass. For the Yang-Mills theory, the infrared cutoff is best provided by dimensional continuation[10].

In the Abelian model $\Pi_{\mu\nu}(q)$ contains only ordinary quantum electrodynamics graphs at the one and two loop levels so that $R_{\Delta E}$ is clearly finite. On the three loop level, the same is true of all graphs except the two shown in Fig. 1. To show that all the cuts of either Fig. 1a or Fig. 1b sum to a finite contribution to $R_{\Delta E}$, it is convenient to use a dispersive representation for the internal photon propagator rather than explicitly considering each cut. Apart from a zeroth order longitudinal piece, the complete photon propagator is $(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2})\ \frac{1}{k^2+i\varepsilon}\ d(k^2)$, where $d(k^2)$ satisfies the dispersion relation

$$\frac{1}{k^2+i\varepsilon}\,d(k^2/M^2,\ m^2/M^2,\ g(M)) = \frac{1}{k^2+i\varepsilon}\,d(0,\ m^2/M^2,\ g(M)) + \int_0^\infty \frac{d\lambda^2\ \Pi(\lambda^2)}{k^2-\lambda^2+i\varepsilon}\quad . \tag{1}$$

Consider the sum of the cuts of Fig. 1a with the muon phase space restricted by the detector kinematics (ΔE cuts). Using Eq. 1 expanded to second order,

$$\sum_{\Delta E\ \text{cuts}} \Pi_{\mu\nu}^{(1a)} = d^{(2)}(0,\, m^2/M^2,\, g(M)) \times \sum_{\Delta E\ \text{cuts}} ---\bigcirc_{1/k^2}---$$
$$+ \int_0^\infty d\lambda^2\, \Pi^{(2)}(\lambda^2) \times \sum_{\Delta E\ \text{cuts}} ---\bigcirc_{1/k^2-\lambda^2}--- \qquad (2)$$

Since the sum of the ΔE cuts of the two loop graph is finite in the region near $\lambda=0$[6], the possibly infrared divergent part in Eq. 2 is proportional to $d^{(2)}(0,m^2/M^2,g(M))+\int_0 d\lambda^2\Pi^{(2)}(\lambda^2)$. However, since $d(k^2/M^2, m^2/M^2, g(M))$ is renormalized at $k^2=-M^2$, the limit $m\to 0$ exists in each order. Thus from Eq. 1, $d^{(2)}(0,m^2/M^2,g(M))+\int_0 d\lambda^2\Pi^{(2)}(\lambda^2)$ is finite in the limit $m\to 0$. Fig. 1b can be dealt with similarly and therefore $R_{\Delta E}(E/M, \Delta E/M, \mu/M, 0, g(M))$ exists to this order. In next order (4 loops in $\Pi_{\mu\nu}(q)$), subgraphs appear with four or more external photon legs and this dispersive method has to be supplemented.

The extension to the Yang-Mills theory is straightforward. The first class of three loop graphs to consider are those without gluon self couplings or Fadeev-Popov loops. Each of these as well as the one and two loop graphs is electrodynamic-like apart from group theory factors. However, since the detector is color blind, the color sum can be done in each graph before making the various cuts. Then since the sum of the ΔE cuts of each such graph in electrodynamics is finite, the same will be true here[11].

The next class contains the graphs with gluon propagator corrections to the two loop graphs. They can be dealt with as in the Abelian model using a dispersive representation for the gluon propagator. The sum of the one loop corrections to the gluon propagator (gluon loop, Fadeev-Popov ghost loop and tadpole) is proportional to $g^2(M)\frac{1}{k^2}\log -k^2/M^2$. This expression can be obtained by dimensional continuation and subtraction at $k^2=-M^2$. A dispersive representation for the gluon propagator to this or any order can be developed by starting with a contour in the complex k^2 plane which comes in from $+\infty$ below the real axis, circles around the origin and goes out to $+\infty$ above the real axis. The radius δ of the small circle around the origin plays the role of the electron mass in the Abelian model and the contribution from the small circle takes the form of the first term in Eq. 1. The sum of this piece and the other is finite as $\delta \to 0$ and the analysis of $\Pi_{\mu\nu}(q)$ then goes as in the Abelian case.

There remain the two diagrams of Fig. 2. We sketch the proof of infrared finiteness for the sum of the cuts of diagram 2b (the Mercedes-Benz diagram). There are five distinct cut diagrams plus their complex conjugates and in each case, the infrared divergences come from various regions of the k and r integrations. It is best to first perform the k_0 and r_0 integrations and then examine each term for divergences in the $\vec{k}$ and $\vec{r}$ integrations. The infrared divergences can be regulated by continuation from three to n-1 spatial dimensions. For the cut diagrams with two real gluons, the energy integrations are trivial. For the other cut diagrams, either one or both of the k_0 and r_0 integrations must be done explicitly by closing the contour in, say, the lpwer half plane and enumerating the poles in that plane.

Divergences can arise from the following regions of $\vec{k}$, $\vec{r}$ space. 1) $\vec{k}$ and $\vec{r} \to 0$, 2) $\vec{k} \to 0$ with $\vec{r}$ fixed, 3) $\vec{r} \to 0$ with $\vec{k}$ fixed, 4) $\vec{k} + \vec{r} \to 0$ with $\vec{k} - \vec{r}$ fixed, 5) $\vec{k}$ becoming parallel to $\vec{r}$ with both vectors non-zero. The last kind of divergence arises only in theories with coupled massless fields. Each of these infrared divergences then cancel among the various terms independently of the quark phase space integration. The treatment of diagram 2a is simpler since the contribution from the region k_μ and $r_\mu \to 0$ vanishes due to the total antisymmetry of the trigluon vertex tensor. We conclude that $R_{\Delta E}$ is infrared finite to this order in the Yang-Mills theory.

The analysis outlined here can be applied to scattering problems as well. The scattering of a color averaged quark beam by an external color singlet field followed by color blind detection is closely analogous to the production process discussed above. We have shown it to be infrared finite to order $g^4(M)$ (two loop corrections to the elastic amplitude and up to two undetected massless quanta in the final state). It should be possible to treat other scattering problems involving color singlet sources and detectors in the same way[12].

We conclude with some comments to help put this work in perspective.

1. Our result, being perturbative, has nothing to do with renormalization group considerations such as asymptotic freedom. $R_{\Delta E}$ is infrared finite in perturbation theory for both the asymptotically free Yang-Mills theory and the non-asymptotically free Abelian model. However, the behavior of $R_{\Delta E}$ to all orders in $g(M)$ is surely quite different in the two cases. One question related to this all-orders behavior is the dependence of $R_{\Delta E}$ on the energy resolution ΔE. In ordinary quantum electrodynamics, the factors of $(\log \Delta E)^n$ which appear in n'th order can be extracted into a multiplicative exponential factor which then vanishes as a power of ΔE when $\Delta E \to 0$. Does something like this happen with coupled massless fields? It is probably not too difficult to answer this question in the Abelian model (model 1) because it becomes weakly coupled in the infrared limit. In the Yang-Mills theory, on the other hand, this is a strong coupling problem.

2. To summarize, our result is a statement about the behavior of $R_{\Delta E}$ order by order in $g(M)$. Since the exact infrared behavior of the Green's functions is unknown in the Yang-Mills theory, $R_{\Delta E}$ has not been expressed in terms of an on-shell coupling constant. Thus, for

example, there is no immediate classical correspondence as in quantum electrodynamics.

3. In two recent papers, Cornwall and Tiktopoulos have suggested that the Bloch-Nordsieck program could fail for Yang-Mills theories resulting in a zero production probability by a color singlet source[13]. This speculation is based on a summation of leading logarithmic corrections to each exclusive emission process. Thus, there is no a priori contradiction with our result which is an order by order statement.

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Footnotes and References

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5. In either theory, the coupling constant g(M) can be defined at some Euclidean scale M. In the Yang-Mills theory, this convention introduces gauge dependence in g at the two loop level. This does not affect our analysis in any order since this procedure leads to an infrared finite coupling constant.
6. The power counting analysis leading to graph-by-graph infrared finiteness applies to any of the usual covariant gauges $D_{\mu\nu}(k)=\frac{1}{k^2}(g_{\mu\nu}+\alpha\frac{k_\mu k_\nu}{k^2})$.
7. T. Kinoshita and A. Sirlin, Phys. Rev. 113, 1652 (1959).
8. This has no doubt been done by many people. Among them are the present authors as well as Dr. A. Ukawa. We thank Prof. T. Kinoshita for telling us of Dr. Ukawa's work. A proof to all orders for Euclidean momenta has recently been completed by E. Poggio and H. Quinn. (E. Poggio, private communication).
9. The angular dependence of $R_{\Delta E}$ is suppressed.
10. The technique of dimensional continuation has been applied to infrared photonic divergences in QED by several people. R. Gastmans and R. Meuldermans, Nucl. Phys. B63, 277 (1973); W.J. Marciano and A. Sirlin, Nucl. Phys. B88, 86 (1975); G. Marques and N. Pananicolaou, Phys. Rev. D (1975), to appear.

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11. The infrared finiteness of the sum of ΔE cuts of each QED graph is a corollary of the Kinoshita analysis [Ref. 3] and is straightforward to verify.

12. Yao has demonstrated the infrared finiteness of quark-quark scattering to lowest order (one loop corrections to the elastic process and up to one undetected soft gluon in the final state). (Y.P. Yao, private communication)

13. J.M. Cornwall and G. Tiktopoulos, Phys. Rev. Lett. 35, 338 (1975) and UCLA preprint UCLA/75/TEP/21, October 1975.

Figure Captions

Fig. 1. Abelian model three loop contributions to $\Pi_{\mu\nu}(q)$, containing coupled massless fields.

Fig. 2. Yang-Mills three loop contributions to $\Pi_{\mu\nu}(q)$.

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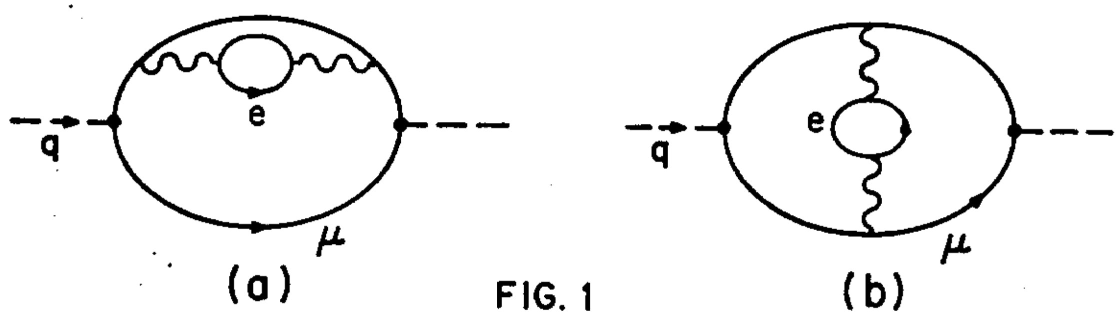

FIG. 1

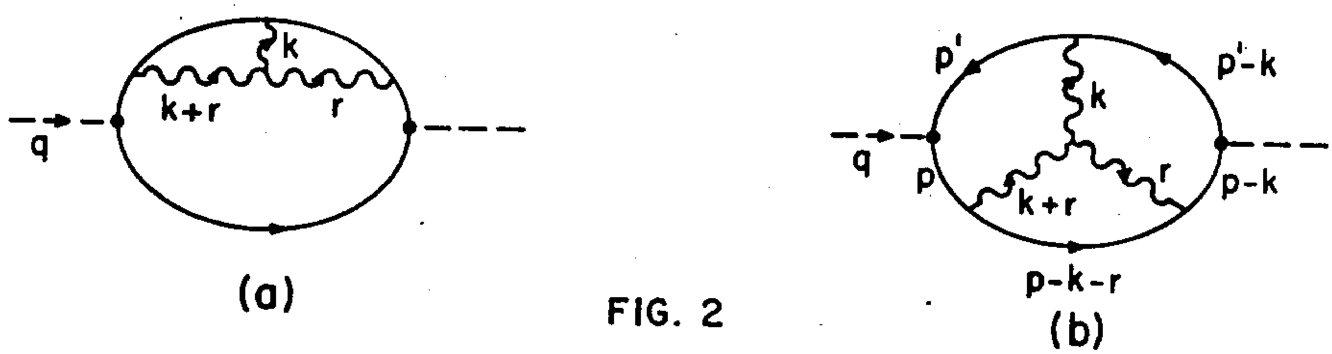

FIG. 2